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Riemann–Liouville fractional integration and reduced distributions on hyperspheres

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Abstract. Using properties of Dirichlet's iterated integral formula we show how the Riemann–Liouville fractional integral unifies arbitrary moment calculations for reduced distributions on hyperspheres. A whole class of problems of this is then reduced to readily identifiable integral transforms. The work is applied to quantum inference and connections made to random matrix theory aspects of nuclear physics and quantum chaos.

1. Introduction

Here we shall develop further an integration formula from previous work [1] and show how it reduces to a special example of the Riemann–Liouville fractional integral [2-4]. The work has ready application to any calculation of the form

$$\int f(\langle\psi|\hat{P}|\psi\rangle)\,\mathrm{d}\hat{\Omega}_{\hat{\psi}} \tag{1}$$

where operator $\hat{P}$ will generally be a *projector*, ψ is a real, complex or quaternionic vector of dimension d and $\mathrm{d}\hat{\Omega}_{\hat{\psi}}$ is the normalized invariant measure over all such vectors. Integrals of this type appear in random matrix theory aspects of nuclear physics [5-7] and quantum chaos [8]. In quantum inference [9] they play a pivotal role in determining the quantum limits to knowledge of states [9].

In [1] we found a useful formula that gave (1) explicitly in terms of various antiderivatives of f for the special case of ψ complex and $\hat{P}$ one-dimensional. In seeking to understand and extend this result we discovered that it is but a particular example of Cauchy's formula for iterated integrals [2]. Here we develop a more general formula for (1) that expresses the result in terms of the Riemann–Liouville fractional antiderivatives of f. In this way we are able to demonstrate how fractional integration unifies all such calculations. A secondary theme shall be the intimate connection of fractional integration with Dirichlet's integral. We find that problems like (1) can be viewed in two quite different ways. The results we then use to calculate some integrals of topical interest.

The paper is organized as follows. In section 2 we review fractional integration and discuss its relation with Dirichlet's integral. In section 3 we use the resulting ideas to arrive at a very economical rederivation of the distribution of components of random unit vectors in $\mathbb{R}^n$. Thus equipped, section 4 develops the connection between these so-called *reduced hyperspherical distributions*, integrals of type (1) and fractional integration. Applications are then developed in section 5.

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2. Riemann–Liouville fractional integration

The Riemann–Liouville fractional integral of function f to the order μ is defined by the expression

$$\mathcal{R}_\mu\{f(w); u\} \equiv \frac{1}{\Gamma(\mu)} \int_0^u f(w)(u-w)^{\mu-1}\, \mathrm{d}w \tag{2}$$

where Re $\mu > 0$ (extensions are possible). It expresses antidifferentiation to fractional degree. To see why, observe that, for Re $\nu > 0$,

$$\mathcal{R}_\mu\{w^{\nu-1}; u\} \equiv \frac{1}{\Gamma(\mu)} \int_0^u w^{\nu-1}(u-w)^{\mu-1}\, \mathrm{d}w = \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)} u^{\nu+\mu-1}$$

where we have recognized Euler's beta function, $B(\mu, \nu)$, at an intermediate step. Now for the μ integral one sees that the result agrees with the usual notion of antiderivative (with constants ignored). Furthermore, for this choice of f it is clear that the μ antiderivative followed by the μ' antiderivative is the $\mu+\mu'$ antiderivative. This essential consistency property ensures that normal integration is recovered should one choose to pass in an arbitrary manner through successive fractional orders to any integral order.

2.1. *Dirichlet's trick*

Clearly the consistency property under iterated fractional integration extends to any f analytic on a disk containing $(0, u)$, for we may then integrate its power series term by term. However, one can do better than that using Dirichlet's integral identity:

$$\int_0^u \int_0^{u-w_2} f(u-w_1-w_2) w_1^{\mu_1-1} w_2^{\mu_2-1}\, \mathrm{d}w_1\, \mathrm{d}w_2$$
$$= \frac{\Gamma(\mu_1)\Gamma(\mu_2)}{\Gamma(\mu_1+\mu_2)} \int_0^u f(u-w) w^{\mu_1+\mu_2-1}\, \mathrm{d}w \tag{3}$$

which we give in a form adapted from [10]. For this to hold f need only be continous on $(0, u)$. Repeated changes of variables to $w_1' = (u-w_2) - w_1$ then $w_2' = u - w_2$ on the left and $w' = u - w$ on the right gives

$$\frac{1}{\Gamma(\mu_1)} \frac{1}{\Gamma(\mu_2)} \int_0^u \int_0^{w_2'} (u-w_2')^{\mu_2-1} (w_2'-w_1')^{\mu_1-1} f(w_1')\, \mathrm{d}w_1'\, \mathrm{d}w_2'$$
$$= \frac{1}{\Gamma(\mu_1+\mu_2)} \int_0^u (u-w')^{\mu_1+\mu_2-1} f(w')\, \mathrm{d}w' \tag{4}$$

thereby proving the desired result

$$\mathcal{R}_{\mu_2}\{\mathcal{R}_{\mu_1}\{f(w_1'); w_2'\}; u\} = \mathcal{R}_{\mu_1+\mu_2}\{f(w'); u\}. \tag{5}$$

Further interesting properties of the fractional integral, such as fractional integration by parts, fractional differentiation and peculiar properties of the fractional chain and Leibniz rules are well documented in the literature [2].

2.2. *Historical note*

The history of this formal extension to the operations of calculus is as old as the subject itself (dating back to Leibniz, see [11]), has had an outstanding pedigree of contributors, and its fair share of colourful controversies (for a detailed survey see [2]). In mathematical physics Riesz used the idea to great effect in developing the theory of hyperbolic partial differential equations [12], whilst it has enjoyed sporadic application in the theory of diffusion (see [2] and references therein) and, in a form due to Weyl [3], formed the basis for Mandelbrot and van Ness's ideas on fractional Brownian motion [11, 13].

For all this the topic appears to remain obscure to all but those who stumble across it. Fundamentally, this is because it is clear from the definition (2) that any result couched in fractional integral terms can always be expressed via ordinary integrals and/or derivatives. Nevertheless, the concept is extremely useful for the different perspective it offers.

2.3. *Dirichlet's integral as a fractional integral*

Dirichlet's integral [10] is a powerful result ideally suited to calculations like (1). We shall give it in a form that reveals most closely the connection with fractional integration. Using (3) repeatedly one readily shows that

$$J \equiv \int \ldots \int_V f(u - w_1 - \ldots - w_d))w_1^{\mu_1 - 1} \ldots w_d^{\mu_d - 1}\,\mathrm{d}w_d \ldots \mathrm{d}w_1$$

$$= \frac{\Gamma(\mu_1)\ldots\Gamma(\mu_d)}{\Gamma(\mu_1 + \ldots + \mu_d)} \int_0^u (u-w)^{|\mu|-1} f(w)\,\mathrm{d}w \tag{6}$$

where V is the simplex; $w_j \geqslant 0$, $\Sigma_{j=1}^d w_j \leqslant u$, $\operatorname{Re}\mu_j > 0$ and $|\mu| = \Sigma_{j=1}^d \mu_j$. Reading the right-hand side as a fractional integral we see that

$$J = \Gamma(\mu_1)\ldots\Gamma(\mu_d) \times \mathcal{R}_{|\mu|}\{f(w);\, u\}. \tag{7}$$

So Dirichlet's integral and the properties of fractional integrals are closely related through their mutual dependence on the identity (3). Indeed we can carry out an entertaining direct proof of Dirichlet's integral using (3) in the form (5). Start by writing the multiple integral as

$$\int_0^{\alpha_1} \ldots \int_0^{\alpha_d} f(\alpha_{d+1}) w_1^{\mu-1} \ldots w_d^{\mu_d - 1}\,\mathrm{d}w_d \ldots \mathrm{d}w_1 \tag{8}$$

where the w_j dependent integration limits are defined recursively by $\alpha_{j+1} = \alpha_j - w_j$ with $\alpha_1 = u$, and it is emphasized that the integrations are to be done in decreasing order of j. An inductive proof of (7) is then readily constructed using the simple property

$$\int_0^{\alpha_j} \mathcal{R}_\mu(f(w);\, \alpha_{j+1}\} w_j^{\mu_j - 1}\,\mathrm{d}w_j = \int_0^{\alpha_j} \mathcal{R}_\mu\{f(w);\, \alpha_j - w_j\} w_j^{\mu_j - 1}\,\mathrm{d}w_j$$

$$= \int_0^{\alpha_j} \mathcal{R}_\mu\{f(w);\, w_j\}(\alpha_j - w_j)^{\mu_j - 1}\,\mathrm{d}w_j$$

$$= \Gamma(\mu_j)\mathcal{R}_{\mu+\mu_j}\{f(w);\, \alpha_j\}.$$

Thus we arrive at an appealing intuitive feel for the meaning of Dirichlet's integral. Note that when all $\mu_j = 1$ we have a normal iterated integral of Cauchy type [2]. The result we derived in [1] for a special case of (1) can now be recognized as being part of the above family of fractional integration formulae.

3. Reduced hyperspherical distributions

Consider a random unit vector $|\hat{r}\rangle \in \mathbb{R}^d$ and an l-dimensional projector $\hat{P}$. We shall seek the distribution of $u \equiv \langle \hat{r}|\hat{P}|\hat{r}\rangle$. the problem is a simple one in hyperspherical geometry, with physical applications in random matrix theory [5-7]. An alternative derivation of the known result is offered upon consideration of the Dirichlet-like integral,

$$\int \cdots \int_V f(w_1 + \ldots + w_l) w_1^{\mu_1 - 1} \ldots w_d^{\mu_d - 1} \delta(u - w_1 - \ldots - w_d)\, \mathrm{d}w_1 \ldots \mathrm{d}w_d \tag{9}$$

where V now extends over all $w_j \geqslant 0$. Applying the methods of section 2, this readily reduces to

$$\frac{\Gamma(\mu_d) \ldots \Gamma(\mu_1)}{\Gamma(|\mu|_1)\Gamma(|\mu|_2)} \int_0^u f(w)(u - w)^{|\mu|_1 - 1} w^{|\mu|_2 - 1}\, \mathrm{d}w \tag{10}$$

where $|\mu|_1 = \Sigma_{j=l+1}^d \mu_j$ and $|\mu|_2 = \Sigma_{j=1}^l \mu_j$. To obtain the reduced hyperspherical distributions, let $\langle \hat{r}|\hat{P}|\hat{r}\rangle = \Sigma_{j=1}^l v_j^2$. If we use a delta function representation of the measure $\mathrm{d}\hat{\Omega}_{\hat{r}}$ and change all d variables to $w_j = v_j^2$, we see that $u = 1$, $\mu_j = \frac{1}{2}$, and so

$$\int f(\langle \hat{r}|\hat{P}|\hat{r}\rangle)\, \mathrm{d}\hat{\Omega}_{\hat{r}}$$

$$= \frac{1}{\mathcal{N}} \frac{\Gamma(1/2)^d}{\Gamma((d-l)/2)\Gamma(l/2)} \int_0^1 f(w)(1 - w)^{(d-l)/2 - 1} w^{l/2 - 1}\, \mathrm{d}w. \tag{11}$$

Set $f(w) \equiv 1$ to obtain $\mathcal{N} = \Gamma(1/2)^d / \Gamma(d/2)$. We can then pick out the sought-after reduced hyperspherical distribution of order l as the weight function

$$p(u) = \frac{\Gamma(d/2)}{\Gamma((d-l)/2)\Gamma(l/2)} (1 - u)^{(d-l)/2 - 1} u^{l/2 - 1} \tag{12}$$

thereby reproducing the standard result [5-8]. Observe (11) has an alternative interpretation as a fractional integral if we define $g(w) = w^{l/2 - 1} f(w)$.

4. Integration formula

It is now a simple matter to establish an explicit formula for (1). For a unified treatment of the three Hilbert spaces over $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$, we label the dimensionality of the base algebra by $\nu = 1$, 2 or 4 respectively. Then from [14] we have

$$\int f(\langle \psi|\hat{P}|\psi\rangle)\, \mathrm{d}\hat{\Omega}_{\hat{\psi}} = \Gamma(\nu d/2) \int f(\langle \psi|\hat{P}|\psi\rangle) \delta(1 - \langle \psi|\psi\rangle)\, \mathrm{d}\psi\, \mathrm{d}\bar{\psi} \tag{13}$$

where $d\psi\, d\bar{\psi} = \Pi_{j=1}^{d} \Pi_{k=1}^{\nu} \pi^{-1/2} x_{jk}$ and j indexes basis elements $|j\rangle$ of the Hilbert space, while k indexes the units e_k of the base algebra. Let $e_1 = 1$, $e_2 = i$, $e_3 = j$ and $e_4 = k$, then we can cover all three cases by writing a general Hilbert space element as

$$|\psi\rangle = \sum_{j=1}^{d} \sum_{k=1}^{\nu} x_{jk} e_k |j\rangle.$$

If we now use polar coordinates to make the change of variables $w_j = \Sigma_{k=1}^{\nu} x_{jk}^2$, then (13) becomes

$$\frac{\Gamma(\nu d/2)}{\Gamma(\nu/2)^d} \int \ldots \int_V f(w_1 + \ldots + w_l) w_1^{\nu/2-1} \ldots w_d^{\nu/2-1} \delta(1 - w_1 - \ldots - w_d)\, dw_1 \ldots dw_d \qquad (14)$$

where again V is over all $w_j \geq 0$. Then, using (11) we arrive at the general result

$$\int f(\langle\psi|\hat{P}|\psi\rangle)\, d\hat{\Omega}_{\bar{\psi}} = \frac{\Gamma(\nu d/2)}{\Gamma(\nu(d-l)/2)\Gamma(\nu l/2)} \int_0^1 f(w)(1-w)^{\nu(d-l)/2-1} w^{\nu l/2-1}\, dw. \qquad (15)$$

Alternatively, we might write this as a fractional integral,

$$\int f(\langle\psi|\hat{P}|\psi\rangle)\, d\hat{\Omega}_{\bar{\psi}} = \frac{\Gamma(\nu d/2)}{\Gamma(\nu l/2)} \times \mathcal{R}_{\nu(d-l)/2}\{g(w); 1\} \qquad (16)$$

where $g(w) = w^{\nu l/2} f(w)$. Observe that where $\nu(d-l)/2$ is integral we get a formula expressible in terms of normal antiderivatives, see for example [9]. This is always so for complex and quaternionic Hilbert space (since ν is even). To obtain a consistent picture for real Hilbert space it is then quite natural to allow half-order integrals (we were led to consider fractional integration via this route). In this sense, the concept of Riemann–Liouville fractional integral exposes an otherwise obscure unity underlying this class of problems. Furthermore, it helps us streamline calculations since many fractional integrals are tabulated [3].

4.1. Extension to non-projectors

Before moving on to discuss applications we remark that in the complex case the above formula can be easily extended to include any operator, $\hat{A}$, admitting an orthonormal spectral decomposition (for instance it may be Hermitian or unitary). Noting that $\nu = 2$ for complex Hilbert space we find that

$$\int f(\langle\psi|\hat{A}|\psi\rangle)\, d\hat{\Omega}_{\bar{\psi}} = \Gamma(d) \int \ldots \int_V f(\lambda_1 w_1 + \ldots + \lambda_d w_d) \delta(1 - w_1 - \ldots - w_d)\, dw_1 \ldots dw_d \qquad (17)$$

where λ_j are the eigenvalues of $\hat{A}$, initially assumed non-degenerate. Proceeding as in section 2 we define recursive limits $\alpha_{j+1} = \alpha_j - w_j$ with $\alpha_1 = 1$ and an auxiliary set of variables $\beta_{j+1} = \beta_j + \lambda_j w_j$ with $\beta_1 = 0$. Then (17) becomes

$$\int f(\langle\psi|\hat{A}|\psi\rangle)\, d\hat{\Omega}_{\bar{\psi}} = \Gamma(d) \int_0^{\alpha_1} \ldots \int_0^{\alpha_{d-1}} f(\beta_d + \lambda_d \alpha_d)\, dw_{d-1} \ldots dw_1. \qquad (18)$$

Using the simple property

$$\int_0^{\alpha_j} \mathcal{R}_m(f(w); \beta_{j+1} + \lambda\alpha_{j+1}\} \, \mathrm{d}w_j$$

$$= \frac{1}{\lambda - \lambda_j} \mathcal{R}_{m+1}\{f(w); \beta_j + \lambda\alpha_j\} + \frac{1}{\lambda_j - \lambda} \mathcal{R}_{m+1}\{f(w); \beta_j + \lambda_j\alpha_j\}$$

where $\lambda \neq \lambda_j$, one can then derive the remarkable formula

$$\int f(\langle\psi|\hat{A}|\psi\rangle) \, \mathrm{d}\hat{\Omega}_{\tilde{\psi}} = \Gamma(d) \sum_{j=1}^{d} \left(\prod_{k \neq j} \frac{1}{\lambda_j - \lambda_k} \right) \mathcal{R}_{d-1}\{f(w); \lambda_j\}. \tag{19}$$

Note that the degenerate case can be handled via the standard trick of taking limits as eigenvalues approach pairwise; for a recent example see [15]. Using this trick one can change variables and treat the d-dimensional quaternionic case as a *doubly degenerate* $2d$-dimensional complex problem. Alternatively, Davies has shown that one can use (19) as a generating function for the corresponding generalization of Dirichlet's integral where all μ_j are integral [16]. However, it seems that the same integral on real Hilbert space (μ_j non-integral) is far more difficult.

5. Applications

For nuclear physics and quantum chaos; we remark that many results concerning the statistical properties of the eigenvectors of random matrices become much clearer if one works with reduced hyperspherical distributions in the general form,

$$p_\nu(u) = \frac{1}{B(\nu(d-l)/2, \nu l/2)} (1-u)^{\nu(d-l)/2-1} u^{\nu l/2-1}. \tag{20}$$

Immediately one recognizes the beta distribution [17]. Then results such as the asymptotic Porter–Thomas χ^2 approximation [5, 7] (for d large and $l \ll d$) become elementary consequences of the asymptotic properties of the beta distribution. One also recognizes different regimes of behaviour, for if d is large and $l \simeq d$ then the distribution becomes Gaussian with mean l/d and variance $\sigma^2 \simeq 2/\nu d(l/d)(1-l/d)$.

Non-trivial new results are to be found when we apply these methods to neatly dispose of a class of integrals occurring in quantum inference. For details about this formalism for constraining knowledge of quantum states see [1, 9]. In particular we shall calculate

$$S(\beta, \hat{P}) = \int (\langle\psi|\hat{P}|\psi\rangle)^\beta \, \mathrm{d}\hat{\Omega}_{\tilde{\psi}} \tag{21}$$

$$H(\beta, \hat{P}) = \int (\langle\psi|\hat{P}|\psi\rangle)^\beta \log\langle\psi|\hat{P}|\psi\rangle \, \mathrm{d}\hat{\Omega}_{\tilde{\psi}} \tag{22}$$

$$E(\beta, \hat{P}) = \int (\langle\psi|\hat{P}|\psi\rangle)^\beta \exp[\alpha\langle\psi|\hat{P}|\psi\rangle] \, \mathrm{d}\hat{\Omega}_{\tilde{\psi}}. \tag{23}$$

Using (3), the formula (16), and the simple results [3]

$$\mathcal{R}_\mu\{w^{\nu-1} \log w, u\} = \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)} u^{\mu+\nu-1} \times [\log u + \Psi(\nu) - \Psi(\mu+\nu)] \tag{24}$$

$$\mathcal{R}_\mu\{w^{\nu-1} \mathrm{e}^{\alpha w}, u\} = \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)} u^{\mu+\nu-1} {}_1F_1(\nu; \mu+\nu; \alpha u) \tag{25}$$

we find

$$S(\beta, \hat{P}) = C(d, l, \nu, \beta) \tag{26}$$

$$H(\beta, \hat{P}) = C(d, l, \nu, \beta) \times [\Psi(\nu l/2+\beta) - \Psi(\nu d/2+\beta)] \tag{27}$$

$$E(\beta, \hat{P}) = C(d, l, \nu, \beta) \times {}_1F_1(\nu l/2+\beta; \nu d/2+\beta; \alpha) \tag{28}$$

where $C(d, l, \nu, \beta)$ is the constant

$$\frac{\Gamma(\nu d/2)}{\Gamma(\nu l/2)} \frac{\Gamma(\nu l/2+\beta)}{\Gamma(\nu d/2+\beta)}.$$

Knowing the above integrals enables us to extend the quantum limits to knowledge of states derived in [9], to the cases of real and quaternionic Hilbert spaces of arbitrary dimension d. This we shall do elsewhere.

Note that use of (19) enables extension to the case where $\hat{P}$ is replaced by $\hat{A}$ complex or quaternionic, Hermitian or unitary. The most significant of these is $H(1, \hat{A})$ on complex Hilbert space. Using (19) we easily find

$$\int \langle\psi|\hat{A}|\psi\rangle \log\langle\psi|\hat{A}|\psi\rangle \, \mathrm{d}\hat{\Omega}_{\bar{\psi}}$$

$$= \Gamma(d) \sum_{j=1}^{d} \left(\prod_{k \neq j} \frac{1}{\lambda_j - \lambda_k} \right) \lambda_j^d [\log \lambda_j + \Psi(2) - \Psi(d+1)] \tag{29}$$

a result first obtained by different means in [15]. This and the corresponding quaternionic result enable a significant generalization of quantum inference to embrace analysers constructed with elements of arbitrary POV measure; see [1] for background.

6. Conclusion

It was our intention both to advertise fractional integration and to develop its use as a tool for a certain class of problems of some topical interest. We hope at least to have shown how knowledge of the concept reveals connections between certain otherwise unrelated topics. The alert reader will have observed that in no sense is the concept necessary for anything that we have done. Nevertheless, we rate it highly useful for the unity it brings to calculations like (1). Furthermore, in making the connection we are directed naturally to the useful resource [3], where the results for a very large class of integrands are tabulated.

Acknowledgments

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